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LETTER TO THE EDITOR

On the non-local generalised symmetries of the Benny–Kaup system

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Abstract. We have extended the formalism of Vinogradov and Krasil'shchik for the coupled system of Benny-Kaup equations. It is explicitly demonstrated that it is possible to generate new non-local symmetries for this coupled equation if we consider $D^{-1}u = \int u \, dx$ to be also in the space of independent variables.

The present theory of infinitesimal symmetries of non-linear partial differential equations considers every higher symmetry of an equation G to be a special type of differentiation of a ring of functions on the space of G, of the infinite extension of G [1]. However, such a concept is too narrow as the recursion operator of these classes of symmetries itself involves the component $\mathrm{D}^{-1}u = \int u \, \mathrm{d}x$, not contained in the original ring of functions [2]. Thus a new kind of symmetries incorporating such kinds of 'non-local' components was considered by Vinogradov and Krasel'shchik [3]. They considered the simplest non-linear system—the Burger equation. Here we show how one can extend this formalism to the case of coupled non-linear systems by considering the Benny-Kaup equations [4]. It may be noted that a very similar system known as dispersive water-wave equations was considered by Kupershmidt [5], who invented a tri-Hamiltonian system for it, and that one can always construct an infinite hierarchy of symmetries by starting with the translational symmetry and the recursion operator. But none of these symmetries belongs to the general class considered here. Actually the need for such an extension was felt long ago.

By analogy with local symmetries which are defined starting from the fact that solutions of the equation G must be transformed into one another, a non-local symmetry is a diffeomorphism of the manifold G that preserves the solution. In this case the base space is equipped with coordinates $(x, t, u_1, u_2, \ldots, v_1, v_2, \mathrm{D}^{-1}u, \mathrm{D}^{-1}v, \ldots)$ where $u_i = \partial^i u/\partial x^i$ and $v_i = \partial^i v/\partial x^i$, and the non-local variables are $\mathrm{D}^{-1}u = \int u \, \mathrm{d}x$ and $\mathrm{D}^{-1}v = \int v \, \mathrm{d}x$. We now define the total x derivative as

$$\mathrm{D}_x = \frac{\partial}{\partial x} + \sum_{i=-1}^{\infty} (u_{i+1}) \frac{\partial}{\partial u_i} + \sum_{i=-1}^{\infty} (h_{i+1}) \frac{\partial}{\partial h_i}$$

and introduce the operator matrix

$$\ni(H, K) = \begin{pmatrix} \sum \dfrac{\partial H}{\partial u_i} \mathrm{D}^i & \sum \dfrac{\partial H}{\partial h_i} \mathrm{D}^i \\ \sum \dfrac{\partial K}{\partial u_i} \mathrm{D}^i & \sum \dfrac{\partial K}{\partial h_i} \mathrm{D}^i \end{pmatrix}$$

where the coupled evolution equations are written as

$$u_t = H(u_i, h_i, x, t) \qquad h_t = K(u_i, h_i, x, t).$$

The Benney-Kaup system is written as

$$u_t = \alpha u_{xx} + uu_x + h_x$$

$$h_t = (hu)_x - \alpha h_{xx}.$$

The case $\alpha = 1$ has been called the dispersive water-wave system. A Lie-Bäcklund symmetry is a transformation:

$$u \to u + \varepsilon\eta^1(u_i, h_i, x, t)$$

$$h \to h + \varepsilon\eta^2(u_i, h_i, x, t)$$

such that (η^1, η^2) satisfies

$$\mathrm{D}_t\eta^1 = \mathrm{D}_{xx}\eta^1 + \mathrm{D}_x\eta^2 + u\mathrm{D}_x(\eta^1 + \eta^1 u_1).$$

$$\mathrm{D}_t\eta^2 = -\mathrm{D}_{xx}\eta^2 + \mathrm{D}_x(h\eta^1 + u\eta^2)$$

where D_t is the total time-derivative operator. In the computation of η^1, η^2 of different order we follow a scheme somewhat different from that of Ibragimov and Anderson [6]. Consider two symmetry generators η^1, η^2 and σ^1, σ^2, then the Jacobi bracket as defined in [3] is not extended by the following formula:

$$\left\{\begin{bmatrix}\eta^1\\ \eta^2\end{bmatrix}, \begin{bmatrix}\sigma^1\\ \sigma^2\end{bmatrix}\right\} = \ni(\eta^1, \eta^2)\begin{pmatrix}\sigma^1\\ \sigma^2\end{pmatrix} - \ni(\sigma^1, \sigma^2)\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}.$$

When one has determined the leading-order term of a particular higher-order symmetry generator and some (x, t)-dependent generator is known, then from the condition that the Jacobi bracket will also be a symmetry we can obtain good information about their formation.

In the present case except for the standard space and time translation symmetry, we have seen that we have, in third order,

$$\eta^1 = u_3 + \tfrac{3}{2}u_2 u + \tfrac{3}{2}h_1 u + \tfrac{3}{2}u_1^2 + \tfrac{3}{4}u^2 u_1 + \tfrac{3}{2}u_1 h$$

$$\eta^2 = h_3 - \tfrac{3}{2}h_2 u + \tfrac{3}{2}uu_1 h - \tfrac{3}{2}h_1 u_1 + \tfrac{3}{4}u^2 h_1 + \tfrac{3}{2}h_1 h.$$

On the other hand we have the Galien symmetry generator,

$$\eta^1 = tu_x + 1$$

$$\eta^2 = th_x$$

which suggests that we search for a generator of the form

$$\eta^1 = tu_{xx} + \text{lower-order terms}$$

$$\eta^2 = -th_{xx} + \text{lower-order terms}.$$

A detailed calculation leads to

$$\eta^1_2 = \tfrac{1}{2}xu_x + t(u_{xx} + h_x + uu_x) + \tfrac{1}{2}u$$

$$\eta^2_2 = \tfrac{1}{2}xh_x + t(-h_{xx} + uh_x + hu_x) + h.$$

The explicit appearance of the non-local operator is observed in third-order generators, which are

$$\eta_3^1 = \tfrac{1}{2}x(u_{xx} + h_x + uu_x) + t(u_{xxx} + \tfrac{3}{2}u_{xx}u + \tfrac{3}{2}u_x^2 + \tfrac{3}{2}hu_x + \tfrac{3}{2}h_x u + \tfrac{3}{4}u^2 u_x)$$
$$+ u_x + \tfrac{1}{4}u_x \mathrm{D}^{-1}u - \tfrac{1}{4}uu_x + \tfrac{1}{4}u^2 + h$$
$$\eta_3^2 = \tfrac{1}{2}x(-h_{xx} + uh_x + hu_x) + t(h_{xxx} - \tfrac{3}{2}h_{xx}u + \tfrac{3}{2}huu_x - \tfrac{3}{2}h_x u_x + \tfrac{3}{4}u^2 h_x + \tfrac{3}{2}h_x h)$$
$$- \tfrac{3}{2}h_x + uh - \tfrac{1}{4}h_x u + \tfrac{1}{4}h_x \mathrm{D}^{-1}u.$$

It is interesting to note that a somewhat similar phenomenon was explored by Kosman-Schwarzbach [7] for the case of ordinary differential equations and the Burger equation.

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